LOCAL POLYNOMIAL CONVEXITY OF THE UNION OF TWO TOTALLY-REAL SURFACES AT THEIR INTERSECTION

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Abstract. We consider the following question: Let S_1 and S_2 be two smooth, totally-real surfaces in $\mathbb{C}^2$ that contain the origin. If the union of their tangent planes is locally polynomially convex at the origin, then is $S_1 \cup S_2$ locally polynomially convex at the origin? If $T_0S_1 \cap T_0S_2 = \{0\}$, then it is a folk result that the answer is *yes*. We discuss an obstruction to the presumed proof, and provide a different approach. When $dim_{\mathbb{R}}(T_0S_1 \cap T_0S_2) = 1$, we present a geometric condition under which no consistent answer to the above question exists. We then discuss conditions under which we can expect local polynomial convexity.

1. Introduction and Statement of Results

The aim of this paper is to provide an answer to the following question:

($*$) *Let S_1 and S_2 be two smooth, totally-real surfaces in $\mathbb{C}^2$ that contain the origin. If the union of their tangent planes is locally polynomially convex at the origin, then is $S_1 \cup S_2$ locally polynomially convex at the origin?*

Our interest is to provide a **complete** analysis of the situation. We were motivated by the following circumstances — which will explain our emphasis on the word "complete" — to discuss the question ($*$).

1) Let S_1 and S_2 be as above. When $T_0S_1 \cap T_0S_2 = \{0\}$, the problem is no doubt familiar to the experts. In this case, the answer to ($*$) is expected to be in the affirmative. The proof, it is asserted, follows from a slight modification of an argument given by Forstnerič and Stout in [4]. While this will work *for most pairs* (S_1, S_2) *in* $\mathbb{C}^2$ (in a sense that will be explained below) it is not clear if such an approach will work universally. *The reader is urged to look at the discussion that immediately follows this list.*
2) It turns out that when T_0S_1 and T_0S_2 contain a line, then $T_0S_1 \cup T_0S_2$ is always locally polynomially convex at the origin. There are some partial answers to ($*$) when $dim_{\mathbb{R}}(T_0S_1 \cap T_0S_2) = 1$; see, for instance, [3]. However, many of the results that we are aware of require S_1 and S_2 to be *real-analytic* surfaces (and one of these results contains an error; see Remark 1.6). In contrast, we wish to answer ($*$) when S_1 and S_2 are merely $\mathcal{C}^k$-smooth, $k \geq 2$.
3) It turns out that, under a certain natural geometric condition, there is no consistent answer to ($*$) when $dim_{\mathbb{R}}(T_0S_1 \cap T_0S_2) = 1$. We would like to demonstrate rigorously what this means, and also to give some conditions under which $S_1 \cup S_2$ is locally polynomially convex at the origin.

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Let us first consider $(*)$ in the case when $T_0S_1 \cap T_0S_2 = \{0\}$. It has been asserted that the proof of the fact that the answer to $(*)$ is, "Yes," is implicit in [4]. Such a proof would go as follows:

- *Step 1.* Show that there is an invertible $\mathbb{C}$-linear transformation that transforms $T_0S_1 \cup T_0S_2$ to $M_1 \cup M_2$, where M_1 and M_2 are totally-real planes of the form

$$(**) \qquad \begin{cases} M_1 \ : \ w = \overline{z} \\ M_2 \ : \ w = r\overline{z} + \varrho z, \quad r \neq 0, \ (r, \varrho) \in \mathbb{R}^2 \setminus \{(1,0)\} \end{cases}$$

- *Step 2.* Use the fact that $T_0S_1 \cup T_0S_2$ is locally polynomially convex at 0 and apply Kallin's Lemma in a similar manner as in [4] to infer that $S_1 \cup S_2$ is locally polynomially convex at 0.

The reason we require M_1 and M_2 to have the form $(**)$ is because *there seems to be no simple way to deduce the desired result via Kallin's Lemma unless r and ϱ in $(**)$ are real*. While the transformation described in Step 1 is possible for most pairs of transverse totally real planes (whose union is locally polynomially convex at the origin) representing (T_0S_1, T_0S_2), we must also contend with the following:

Observation 1.1. *There is at least one one-parameter family of linear transformations $\{S_p : p \in \mathbb{R} \setminus \{0\}\}$ of $\mathbb{C}^2$ such that*

$$(S_p + i\mathbb{I})(\mathbb{R}^2) \text{ is totally real } \forall p \in \mathbb{R} \setminus \{0\},$$
$$(S_p + i\mathbb{I})(\mathbb{R}^2) \cap \mathbb{R}^2 = \{0\} \ \forall p \in \mathbb{R} \setminus \{0\},$$
$$(S_p + i\mathbb{I})(\mathbb{R}^2) \cup \mathbb{R}^2 \text{ is locally polynomially convex at } 0 \ \forall p \in \mathbb{R} \setminus \{0\},$$

but for each $p \in \mathbb{R} \setminus \{0\}$, there exists no invertible $\mathbb{C}$-linear transformation of $\mathbb{C}^2$ that can map $\mathbb{R}^2 \cup (S_p + i\mathbb{I})(\mathbb{R}^2)$ to a union $M_1 \cup M_2$ with (M_1, M_2) having the form $(**)$.

The details of the above are presented in sub-section 2.1. We do not doubt that the above two-step approach *could be made to work even when r and ϱ in $(**)$ take non-real values*, but this would require at least a more sophisticated Kallin polynomial (hence much harder calculations) and may, perhaps, even require some further inputs besides those in [4]. Consequently, we try another approach by modifying some ideas of Weinstock — which enables us to deal with $T_0S_1 \cup T_0S_2$ without having to transform the planes to graphs — to get Theorem 1.2 below. The latter method has the advantage that it is more readily adapted to the problem of studying local polynomial convexity at $0 \in \mathbb{C}^2$ of the union of *more than two* totally-real planes in $\mathbb{C}^2$ intersecting at 0. The latter problem is of some interest because it provides the means to investigate local polynomial convexity of a smooth real surface $S \subset \mathbb{C}^2$ at a point $p \in S$ at which T_pS is a complex line. This general principle was, in fact, introduced in [4]. In general — as the papers [1] and [2] reveal — detecting local polynomial convexity at a degenerate "non-parabolic" complex-tangency would require the study of the union of more than two totally-real surfaces, intersecting transversely at $0 \in \mathbb{C}^2$. These issues will be tackled in a different article. With this background, we can announce:

Theorem 1.2. *The union of two $\mathcal{C}^2$-smooth totally-real surfaces in $\mathbb{C}^2$ intersecting transversally only at the origin is locally polynomially convex if the union of their tangent spaces at the origin is locally polynomially convex at the origin.*

Weinstock [7] gave a criterion for the union of two transverse, maximally totally-real subspaces in $\mathbb{C}^n$ to be locally polynomially convex at the origin. Our proof of Theorem 1.2 relies upon a normal form, developed in [7], for a pair of totally-real planes intersecting transversely at $0 \in \mathbb{C}^2$, and on Kallin's Lemma (see Lemma 3.1 below). We note that the condition stated in $(*)$ *cannot* be necessary and sufficient; see [7, Example 5].

The following lemma is essential in setting the context for the next three theorems.

Lemma 1.3. *Let M_j, $j = 1, 2$, be two distinct totally-real planes in $\mathbb{C}^2$ containing the origin, such that $dim_{\mathbb{R}}(M_1 \cap M_2) = 1$. Then $M_1 \cup M_2$ is locally polynomially convex at the origin.*

We shall *not* give a separate proof for the above; the proof will follow along similar lines as the proof of Theorem 1.5 below. This lemma establishes that question $(*)$ remains valid when $dim_{\mathbb{R}}(M_1 \cap M_2) = 1$. Our next theorem shows that the answer to $(*)$ is not always affirmative when $dim_{\mathbb{R}}(T_0S_1 \cap T_0S_2) = 1$. Comparing this with Theorem 1.8 will reveal that that there is no consistent answer to $(*)$ when

(I) $dim_{\mathbb{R}}(T_0S_1 \cap T_0S_2) = 1$; and
(II) $span_{\mathbb{C}}\{T_0S_1 \cap T_0S_2\} \subset\ span_{\mathbb{R}}\{T_0S_1 \cup T_0S_2\}$.

Refer to the remarks following Theorem 1.8 for a clarification of the last assertion.

Before stating Theorem 1.4, we need to define one term. Given a set $S \subset \mathbb{C}^2$, we say that S^ε is an *ε-perturbation* of S if S^ε is the image of S under a $\mathcal{C}^1$-diffeomorphism Θ_ε defined in a neighbourhood U of S such that $\|\mathrm{id}_U - \Theta_\varepsilon\|_{\mathcal{C}^1} \lesssim \varepsilon$.

Theorem 1.4. *Let M_j, $j = 1, 2$, be two distinct totally-real planes in $\mathbb{C}^2$ containing the origin, such that $dim_{\mathbb{R}}(M_1 \cap M_2) = 1$ and $span_{\mathbb{C}}\{M_1 \cap M_2\}$ lies in the real hyperspace that contains $M_1 \cup M_2$. Then for each $\varepsilon > 0$, there exists a $\delta > 0$ and totally-real submanifolds S_j^ε, $j = 1, 2$, of $B(0; 2\delta)$ such that:*

- *$S_j^\varepsilon \cap \overline{B(0;\delta)}$ are ε-perturbations of $M_j \cap \overline{B(0;\delta)}$, $j = 1, 2$,*
- *$T_0S_1^\varepsilon \cup T_0S_2^\varepsilon = M_1 \cup M_2$,*

and such that $S_1^\varepsilon \cup S_2^\varepsilon$ is not polynomially convex.

Here, $B(a; r)$ denotes the Euclidean ball in $\mathbb{C}^2$ with centre at a and radius $r > 0$.

Theorem 1.4 raises the following question: *what can we say if S_1 and S_2 are as in $(*)$, $dim_{\mathbb{R}}(T_0S_1 \cap T_0S_2) = 1$, and $span_{\mathbb{C}}\{T_0S_1 \cap T_0S_2\} \nsubseteq\ span_{\mathbb{R}}\{T_0S_1 \cup T_0S_2\}$?* In response to this question we have the following result:

Theorem 1.5. *Let S_1 and S_2 be two $\mathcal{C}^2$-smooth surfaces in $\mathbb{C}^2$ that contain the origin. Assume that*

- *$dim_{\mathbb{R}}(T_0S_1 \cap T_0S_2) = 1$; and*
- *$span_{\mathbb{C}}\{T_0S_1 \cap T_0S_2\} \nsubseteq\ span_{\mathbb{R}}\{T_0S_1 \cup T_0S_2\}$.*

If $(S_1 \cup S_2) \subset\ span_{\mathbb{R}}\{T_0S_1 \cup T_0S_2\}$, then $S_1 \cup S_2$ is locally polynomially convex at the origin.

Remark 1.6. Unbeknownst to me, Dieu had announced the following result in [3]:

> **Result 1.7** (Prop. 2.2, [3])**.** *Let φ be a real-valued function defined in a neighbourhood of $0 \in \mathbb{C}$ and of class $\mathcal{C}^1$. Define*
>
> $S_1 := \{(z, w) \in \mathbb{C}^2 : w = \overline{z}\}$,
>
> $S_2 := \{(z, w) \in \mathsf{Dom}(\varphi) \times \mathbb{C} : w = (1 + \lambda)\overline{z} + \overline{\lambda} z + \varphi(z)\}$ $(\lambda \neq 0, -1)$.

Then $S_1 \cup S_2$ *is* not *locally polynomially convex at* 0 *if and only if*

i) λ *is real; and*

ii) *For every* t *sufficiently close to* $0 \in \mathbb{R}$, *the set* $\{z \in \mathsf{Dom}(\varphi) : \mathfrak{Re}(z) = t/2,\ 2\lambda\mathfrak{Re}(z) + \varphi(z) = 0\}$ *contains at most one component.*

It was brought to my notice that Theorem 1.5 follows immediately from the above result; or — at any rate — in the generic arrangement of tangents when $T_0S_1 = \{(z,w) \in \mathbb{C}^2 : w = \overline{z}\}$ and $T_0S_2 = \{(z,w) \in \mathbb{C}^2 : w = (1+\lambda)\overline{z} + \overline{\lambda}z\}$ and when S_1 and S_2 are $\mathcal{C}^\omega$-surfaces (in which case S_1 can always be taken as $\{(z,w)\ :\ w = \overline{z}\}$ locally). In this setting, the argument would go as follows:

- The condition in Result 1.7 that φ be real-valued is equivalent to our condition $(S_1 \cup S_2) \subset\ span_{\mathbb{R}}\{T_0S_1 \cup T_0S_2\}$ (in Theorem 1.5); and
- The *negation* of the condition (i) in Result 1.7 is equivalent to our condition $span_{\mathbb{C}}\{T_0S_1 \cap T_0S_2\} \nsubseteq\ span_{\mathbb{R}}\{T_0S_1 \cup T_0S_2\}$.

However, it turns out that *the condition* [(*i*) *AND* (*ii*)] *is neither necessary nor sufficient for* $S_1 \cup S_2$ *to fail to be polynomially convex.* A demonstration of this is presented in sub-section 2.2 below. This observation also demands that we prove Theorem 1.5 from scratch.

We now consider totally-real graphs $S_j,\ j = 1,2$. When $dim_{\mathbb{R}}(T_0S_1 \cap T_0S_2) = 1$, then one expects polynomial convexity to be influenced by the higher-order terms in the graphing functions. This is the intuition behind the next theorem. Given such graphs, it can be shown that there is a global holomorphic change of coordinates with respect to which S_1 and S_2 have the representations given in Theorem 1.8. To reiterate: the representations of the graphs S_1 and S_2 in the first half of Theorem 1.8 *are not simplifying assumptions.*

Theorem 1.8. *Let* $S_j,\ j = 1,2$, *be two* $\mathcal{C}^\infty$*-smooth totally-real surfaces in* $\mathbb{C}^2$ *containing the origin such that* $T_0S_1 \neq T_0S_2$ *and* $T_0S_1 \cap T_0S_2$ *contains a real line. In a neighbourhood* U *of the origin, we present:*

$$S_1 \cap U = \{(z, \overline{z} + \overline{A}z^2 + A\overline{z}^2 + C_1 z\overline{z} + O(|z|^3)) : z \in D(0;\delta)\},$$
$$S_2 \cap U = \{(z, \overline{z} + \overline{\lambda}z + \lambda\overline{z} + \phi_2(z)) : z \in D(0;\delta)\},$$

where $\delta > 0$, $\phi_2 \in \mathcal{C}^\infty(D(0;\delta))$ *and* $\phi_2(z) = A_2z^2 + B_2\overline{z}^2 + C_2 z\overline{z} + O(|z|^3)$.
Suppose:

(*i*) *(Non-degeneracy condition)* $\mathfrak{Im}(C_1) \neq 0$, $\mathfrak{Im}\left(\frac{(\overline{A_2}-B_2)\overline{\lambda}^2}{|\lambda|^2} + C_2\right) \neq 0$ *and have opposite signs;*

(*ii*) $\mathrm{sgn}(\mathfrak{Im}(C_1))\mathfrak{Im}\left((\overline{A_2} - B_2)\overline{\lambda}\right) < \frac{1}{2}\left|\mathfrak{Im}\left(\frac{(\overline{A_2}-B_2)\overline{\lambda}^2}{|\lambda|^2} + C_2\right)\right|$.

Then $S_1 \cup S_2$ *is locally polynomially convex at the origin.*

Remark 1.9. The conditions (*i*) and (*ii*) might look somewhat artificial at first glance, but we formulated them with the following phenomenon in mind. When $\overline{A_2} = B_2$ and $\mathfrak{Im}(C_j) = 0,\ j = 1,2$, then the resulting graphs S_1^0 and S_0^2 are in fact examples of the surfaces discussed in Theorem 1.4. Still keeping $\overline{A_2} = B_2$, we see that if we alter the coefficients C_j slightly so that $\mathfrak{Im}(Cj) = \varepsilon,\ j = 1,2$, then the resulting $S_1 \cup S_2$ is an ε-perturbation of $S_1^0 \cup S_2^0$, and the local hull of $S_1^0 \cup S_2^0$ collapses under the perturbation. Summarizing in a coordinate-free manner: when $dim_{\mathbb{R}}(T_0S_1 \cap T_0S_2) = 1$ and

$$span_{\mathbb{C}}\{T_0S_1 \cap T_0S_2\} \subset\ span_{\mathbb{R}}\{T_0S_1 \cup T_0S_2\},$$

it is possible for $S_1 \cup S_2$ to not be locally polynomially convex at 0 and yet, given any $\varepsilon > 0$, admit ε-perturbations S_j^ε with $T_0 S_j^\varepsilon = T_0 S_j$, $j = 1, 2$, such that $S_1^\varepsilon \cup S_2^\varepsilon$ is locally polynomially convex at the origin. I.e., when the pair (S_1, S_2) has the properties (I) and (II) listed just after Lemma 1.3, then the question $(*)$ has no coherent answer.

A few words about the layout of this paper. We would first like to conclude the technical discussion on the relationship between a couple of theorems and the folk results to which they seem associated. This will be the subject of the next section. Section 3 will elaborate on some technical preliminaries needed in the proofs of our results. The proofs of our four theorems will be found in Sections 4–7.

2. Relations with known results

2.1. **Concerning Observation 1.1.** Consider the two planes: $P_1 := \mathbb{R}^2$ and $P_2 := span_{\mathbb{R}}\{(s,t),(\sigma,\tau)\}$ — $s,t,\sigma,\tau \in \mathbb{C}$ — with $P_1 \cap P_2 = \{0\}$. First, note that if there exists a $\mathbb{C}$−linear, invertible map $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ such that

$$\begin{aligned} &T(P_1) = \{(z,w) \in \mathbb{C}^2 : w = \overline{z}\}, \\ &T(P_2) = \{(z,w) \in \mathbb{C}^2 : w = r\overline{z} + \varrho z\}, \quad r \neq 0, \ (r,\varrho) \in \mathbb{R}^2 \setminus \{(1,0)\}, \end{aligned} \tag{2.1}$$

then T must have the matrix representation M_T (with respect to the standard basis) given by

$$M_T = \begin{pmatrix} A & B \\ \overline{A} & \overline{B} \end{pmatrix},$$

where $A(= \alpha_1 + i\alpha_2)$, $B(= \beta_1 + i\beta_2) \in \mathbb{C}$ and, for invertibility $A\overline{B} \notin \mathbb{R}$. If, however, we interchange the desired images of P_1 and P_2 under T, then T must have the following matrix representation:

$$M_T = \begin{pmatrix} A & B \\ r\overline{A} + \varrho A & r\overline{B} + \varrho B \end{pmatrix}. \tag{2.2}$$

Motivated by Weinstock's work [7], we shall focus on $P_2 (:= span_{\mathbb{R}}\{(s,t),(\sigma,\tau)\})$ determined by

$$\begin{pmatrix} s & \sigma \\ t & \tau \end{pmatrix} = \begin{pmatrix} p+i & 0 \\ 0 & q+i \end{pmatrix}$$

(which gives one of the three normal forms for a pair of totally-real planes in $\mathbb{C}^2$ intersecting transversely at $0 \in \mathbb{C}^2$).

T having the mapping properties given in (2.1) exists (and we will implicitly view the necessary conditions as a linear system with r and ϱ as unknowns):

$$\Rightarrow \begin{cases} \overline{A}(p-i)r + A(p+i)\varrho &= \overline{A}(p+i) \\ \overline{B}(q-i)r + B(q+i)\varrho &= \overline{B}(q+i) \end{cases}$$

has a solution in $\mathbb{R}^2 \setminus ((\{0\} \times \mathbb{R}) \cup \{(1,0)\})$

for some $(A,B) \in \mathbb{C}^2$ *such that* $A\overline{B} \notin \mathbb{R}$.

Considering real and imaginary parts separately, the existence of the desired T

$$\Rightarrow \begin{cases} (\alpha_1 p - \alpha_2)r + (\alpha_1 p - \alpha_2)\varrho &= \alpha_1 p + \alpha_2 \\ -(\alpha_2 p + \alpha_1)r + (\alpha_2 p + \alpha_1)\varrho &= \alpha_1 - \alpha_2 p \\ (\beta_1 p - \beta_2)r + (\beta_1 p - \beta_2)\varrho &= \beta_1 p + \beta_2 \\ -(\beta_2 p + \beta_1)r + (\beta_2 p + \beta_1)\varrho &= \beta_1 - \beta_2 p \end{cases} \tag{2.3}$$
$$\textit{has a solution in } \mathbb{R}^2 \setminus ((\{0\} \times \mathbb{R}) \cup \{(1,0)\})$$
$$\textit{for some } (A,B) \in \mathbb{C}^2 \textit{ such that } A\overline{B} \notin \mathbb{R}.$$

Let us restrict ourselves to $p = q \neq 0$. In this case, if $(\alpha_1 p - \alpha_2) = 0$, then the consistency of the above system of equations forces on us:

$$(\alpha_1 p - \alpha_2) = 0 \text{ and } (\alpha_1 p + \alpha_2) = 0.$$

That implies $\alpha_1 + i\alpha_2 = 0$, which contradicts the invertibility of T. Thus $\alpha_1 p - \alpha_2 \neq 0$. Similarly, all the coefficients of the left hand side of the above system of equations (2.3) are non-zero. Thus, T having the mapping properties given in (2.1) exists

$$\Rightarrow \begin{cases} \dfrac{\alpha_1 p + \alpha_2}{\alpha_1 p - \alpha_2} &= \dfrac{\beta_1 q + \beta_2}{\beta_1 q - \beta_2}, \\ \\ \dfrac{\alpha_1 - \alpha_2 p}{\alpha_1 + \alpha_2 p} &= \dfrac{\beta_1 - \beta_2 q}{\beta_1 + \beta_2 q} \end{cases}$$
$$\Rightarrow \begin{cases} \dfrac{\alpha_2}{\alpha_1 p - \alpha_2} &= \dfrac{\beta_2}{\beta_1 q - \beta_2}, \\ \\ \dfrac{\alpha_1}{\alpha_1 + \alpha_2 p} &= \dfrac{\beta_1}{\beta_1 + \beta_2 q} \end{cases}$$
$$\Rightarrow \begin{cases} \alpha_1\beta_2 - \beta_1\alpha_2 &= 0, \\ \alpha_2\beta_1 - \beta_2\alpha_1 &= 0 \ \ (\text{since } p = q \neq 0). \end{cases}$$

But the second condition implies that $\mathfrak{Im}(A\overline{B}) = 0$, i.e. $A\overline{B} \in \mathbb{R}$, which is a contradiction. Thus there is no T with the mapping properties given in (2.1).

Under the assumption $p = q$, we still need to show that there is no invertible $\mathbb{C}$-linear map that maps $P_1 \cup P_2$ to the union of the two graphs given in (2.1), but with the images swapped. Ruling this out is a shorter argument. In this case, T will have the matrix representation given by (2.2). Hence, T having the desired properties exists (this time, we implicitly view the necessary conditions as a linear system with α_1, α_2, β_1, β_2 as unknowns)

$$\Rightarrow \begin{cases} p(r + \varrho - 1)\alpha_1 + (r - \varrho + 1)\alpha_2 &= 0 \\ (r + \varrho + 1)\alpha_1 + p(\varrho - r + 1)\alpha_2 &= 0 \\ p(r + \varrho - 1)\beta_1 + (r - \varrho + 1)\beta_2 &= 0 \\ (r + \varrho + 1)\beta_1 + p(\varrho - r + 1)\beta_2 &= 0 \end{cases}$$
$$\textit{has a solution in } \mathbb{R}^4 \textit{ such that } \alpha_1\beta_2 - \beta_1\alpha_2 \neq 0$$
$$\textit{for some } (r, \varrho) \in \mathbb{R}^2 \setminus ((\{0\} \times \mathbb{R}) \cup \{(1,0)\}).$$

Hence, every (α_1, α_2) and (β_1, β_2) such that $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ is a solution of the above system will be solutions in $\mathbb{R}^2$ of the following system of equations

$$\begin{aligned} p(r+\varrho-1)X + (r-\varrho+1)Y &= 0 \\ (r+\varrho+1)X + p(\varrho-r+1)Y &= 0. \end{aligned}$$

For the matrix M_T in (2.2) to be invertible, we need $\{(\alpha_1, \alpha_2), (\beta_1, \beta_2)\}$ to be linearly independent in $\mathbb{R}^2$. The only way we can get $\{(\alpha_1, \alpha_2), (\beta_1, \beta_2)\}$ to be linearly independent is for each coefficient of the above system of equations to vanish. This gives $r = 0$, which is a contradiction. Hence a T with the matrix representation (2.2) having the other desired properties cannot exist.

It follows from the work of Weinstock [7] (refer to the last paragraph of Section 3 for a precise statement) that, by our choice of (s, t) and (σ, τ), $P_1 \cup P_2$ is locally polynomially convex. To conclude: it can easily be checked that the transformations S_p determined (with respect to the standard basis) by the matrices

$$\begin{pmatrix} p+i & 0 \\ 0 & p+i \end{pmatrix}, \quad p \in \mathbb{R} \setminus \{0\},$$

give us the 1-parameter family $\{S_p : p \in \mathbb{R} \setminus \{0\}\}$ having all the properties stated in Observation 1.1.

2.2. **A discussion on the correctness of Result 1.7.** Another issue that — as we discussed in Section 1 — needs to be settled is the status of Result 1.7. We address this now. First, we shall show the following: There exist $\lambda \in \mathbb{R} \setminus \{0, -1\}$ and a real-valued function $\varphi \in \mathcal{C}^1(\{0\})$ such that, $S_1 \cup S_2$ in 1.7 *is not polynomially convex at* $(0, 0)$, and yet $\{z \in \mathbb{C} : \mathfrak{Re} z = t/2,\ 2\lambda \mathfrak{Re} z + \varphi(z) = 0\}$ *has more than one connected components* for all $t > 0$. The proof goes as follows:

Let $\varphi(z) = (\mathfrak{Im} z)^2 = y^2$ (writing $z = x + iy$), and consider the polynomial $p(z, w) = z + w$. The polynomial p is real valued when it is restricted to $S_1 \cup S_2$. Let $V_t = p^{-1}\{t\}$. Now let us compute $V_t \cap S_j,\ j = 1, 2$. We have

$$\begin{aligned} V_t \cap S_1 &= \{(z, \overline{z}) : \mathfrak{Re} z = t/2\}, \\ V_t \cap S_2 &= \{(z, \overline{z} + \overline{\lambda} z + \lambda \overline{z}) : 2\mathfrak{Re} z + 2\lambda \mathfrak{Re} z + (\mathfrak{Im} z)^2 = t\}. \end{aligned}$$

Let π_1 denote the projection onto the first coordinate. Then, the above are curves in $\mathbb{C}^2$ that project down to:

$$\begin{aligned} \pi_1(V_t \cap S_1) &= \{(x, y) \in \mathbb{R}^2 \ : \ x = t/2\}, \\ \pi_1(V_t \cap S_2) &= \{(x, y) \in \mathbb{R}^2 \ : \ 2x + 2\lambda x + y^2 = t\}. \\ &= \left\{(x, y)\mathbb{R}^2 \ : \ x - \frac{t}{2(1+\lambda)} = -\frac{y^2}{2(1+\lambda)}\right\}. \end{aligned}$$

Let us now choose $\lambda \ : \ -1 < \lambda < 0$, and fix it. For $t > 0$, we see that:

(1) $\pi_1(V_t \cap S_1) \cap \pi_1(V_t \cap S_2)$ consists of the two points $(t/2, \pm\sqrt{t|\lambda|})$; and
(2) $\mathbb{C} \setminus \pi_1(V_t \cap S_1) \cup \pi_1(V_t \cap S_2)$ contains a bounded component, say $\mathcal{D}_t$, and $\mathcal{D}_t \to \{0\}$ as $0 < t \searrow 0$.

Let us now write $\pi_1(V_t \cap S_1) \cap \pi_1(V_t \cap S_2) = \{\zeta_1(t), \zeta_2(t)\},\ t > 0$. Note:

$$\pi_1^{-1}\{\zeta_j(t)\} \cap S_1 = \{(\zeta_j(t), t - \zeta_j(t))\} = \pi_1^{-1}\{\zeta_j(t)\} \cap S_2,\ j = 1, 2,\ t > 0.$$

From this and (2), we conclude that $V_t \cap (M_1 \cup M_2)$ determines a closed curve C_t such that

$$\pi_1(C_t) = \partial\mathcal{D}_t, \ \forall t \ : \ 0 < t \searrow 0.$$

Hence, we get a family of analytic discs $\psi_t : \mathcal{D}_t \longrightarrow \mathbb{C}^2$ by $z \longmapsto (z, t-z)$, attached to $S_1 \cup S_2$ and $\psi_t \to 0$ as $t \searrow 0$. Hence, by maximum modulus theorem, $S_1 \cup S_2$ is not locally polynomially convex at the origin. Yet, owing to (1),

$$\left\{ z \in \mathbb{C} : \mathfrak{Re} z = \frac{t}{2}, \ 2\lambda \mathfrak{Re} z + (\mathfrak{Im}(z))^2 = 0 \right\}$$

has two connected components for all $t > 0$.

This shows that condition $[(i)$ AND $(ii)]$ in Result 1.7 is not always necessary for $S_1 \cup S_2$ to not be locally polynomially convex at $(0,0) \in \mathbb{C}^2$.

Now we shall show that there exists $\lambda \in \mathbb{R} \setminus \{0, -1\}$ and a real-valued function $\varphi \in \mathcal{C}^1(\{0\})$ such that the condition (ii) is satisfied and yet $S_1 \cup S_2$ in Result 1.7 is locally polynomially convex at $(0,0) \in \mathbb{C}^2$. Let us consider the following surfaces in $\mathbb{C}^2$:

$$\begin{aligned} S_1 &:= \{(z, w) \in \mathbb{C}^2 : w = \overline{z}\}, \\ S_2 &:= \{(z, w) \in D(0;\delta) \times \mathbb{C} : w = (1+\lambda)\overline{z} + \overline{\lambda} z + \varphi(z)\}, \end{aligned}$$

where $\lambda \in \mathbb{R} \setminus \{0, -1\}$, $\delta > 0$ and

$$\varphi = \left. \Phi(\mathfrak{Re}(\cdot)) \right|_{D(0;\delta)},$$

where $\Phi \in \mathbb{R}[x]$, i.e. a polynomial in $x := \mathfrak{Re} z$ with real coefficients, such that $\Phi(0) = 0 = \Phi'(0)$. Let us consider the polynomial $P(z, w) = z + w$. Now let us compute the set $P^{-1}\{t\} \cap S_j$ for $j = 1, 2$.

$$\begin{aligned} P^{-1}\{t\} \cap S_1 &= \{(z, \overline{z}) \in \mathbb{C}^2 : z + \overline{z} = t\} \\ &= \{(t/2, +iy, t/2 - iy) \in \mathbb{C}^2 : y \in \mathbb{R}\} \quad (\text{writing } z = x + iy), \end{aligned} \tag{2.4}$$

$$\begin{aligned} P^{-1}\{t\} \cap S_2 &= \{(z, w) \in D(0;\delta) \times \mathbb{C} : w = (1+\lambda)\overline{z} + \overline{\lambda} z + \varphi(z), \ z + w = t\} \\ &= \{(x + iy, x - iy + 2\lambda x + \varphi(x)) \in D(0;\delta) \times \mathbb{C} : 2(1+\lambda)x + \Phi(x) = t\}. \end{aligned} \tag{2.5}$$

Let $q_t(x) = 2(1+\lambda)x + \Phi(x) - t$ and let $Z_{\mathbb{R}}(q_t)$ denote the set of real zeros of the polynomial q_t. We have:

$$\begin{aligned} &\left\{ z \in D(0;\delta) : \mathfrak{Re} z = \frac{t}{2}, \ 2\lambda \mathfrak{Re} z + \Phi(z) = 0 \right\} \\ &= \begin{cases} \varnothing, & \text{if } t/2 \notin Z_{\mathbb{R}}(q_t), \\ \{(t/2 + iy) : y \in \mathbb{R}\} \cap D(0;\delta), & \text{if } t/2 \in Z_{\mathbb{R}}(q_t). \end{cases} \end{aligned}$$

This shows that if we fix $\delta > 0$ to be sufficiently small then, because $Z_{\mathbb{R}}(q_t) \cap (-\delta, \delta)$ is at most a singleton, the set $\left\{ z \in \mathbb{C} : \mathfrak{Re} z = \frac{t}{2}, \ 2\lambda \mathfrak{Re} z + \varphi(z) = 0 \right\}$ has at most one component, for all $t \in \mathbb{R}$ sufficiently small. Hence the condition (ii) of Result 1.7 holds.

From (2.4) and (2.5), we have the following:

$$
\begin{aligned}
P^{-1}\{t\} \cap ((S_1 \cup S_2) \cap \overline{D(0;\varepsilon)} \times \mathbb{C}) \\
&= \{(t/2, +iy, t/2 - iy) \in \overline{D(0;\varepsilon)} \times \mathbb{C} : y \in \mathbb{R}\} \\
&\quad \cup \left(\cup_{x \in Z_{\mathbb{R}}(q_t) \cap [\varepsilon,\varepsilon]} \{(x+iy, -x-iy+t) : \sqrt{x^2+y^2} \leq \varepsilon\} \right).
\end{aligned}
$$

These are lines segments in $\overline{D(0;\varepsilon)} \times \mathbb{C}$ whose projections on $\mathbb{C}_z$ are line segments parallel to y-axis. Hence, $P^{-1}\{t\} \cap ((S_1 \cup S_2) \cap \overline{D(0;\varepsilon)} \times \mathbb{C})$ is union of finitely many non-intersecting line segments when $\varepsilon \in (0,\delta)$. Hence

$$
(P^{-1}\{t\} \cap ((S_1 \cup S_2) \cap \overline{D(0;\varepsilon)} \times \mathbb{C})\widehat{)} = P^{-1}\{t\} \cap ((S_1 \cup S_2) \cap \overline{D(0;\varepsilon)} \times \mathbb{C}),
$$

for $\varepsilon \in (0,\delta)$ and for all $t \in \mathbb{R}$ sufficiently small. Hence, the pair (S_1, S_2) satisfies the conditions (i) and (ii) in Result 1.7 and yet $S_1 \cup S_2$ is locally polynomially convex at the origin in $\mathbb{C}^2$. This last assertion follows from a very useful result — see Result 3.2 below— for computing polynomial hulls.

This shows that condition $[(i)\ \text{ AND } (ii)]$ in Result 1.7 is not sufficient for $S_1 \cup S_2$ to not be locally polynomially convex at $(0,0) \in \mathbb{C}^2$.

3. Technical preliminaries

We shall require a couple of preliminaries to set the stage for proving the above theorems. The principal tool that we shall use is the following lemma by Kallin [5].

Lemma 3.1 (Kallin)**.** *Let K and L be two compact polynomially convex subsets in $\mathbb{C}^n$. Suppose there exists a holomorphic polynomial P satisfying the following conditions:*

(i) $\widehat{P(K)} \cap \widehat{P(L)} \subseteq \{0\}$ *and* $0 \in \mathbb{C} \setminus int(\widehat{P(K)} \cup \widehat{P(L)})$*; and*
(ii) $P^{-1}\{0\} \cap (K \cup L)$ *is polynomially convex.*

Then $K \cup L$ is polynomially convex.

The other tool we shall use in the course of the proof of some of the above theorems is the following theorem from Stout's book [6, Theorem 1.2.16].

Result 3.2. *If $X \subset \mathbb{C}^n$ is compact and if $\mathcal{P}(X)$ contains a real valued function, say f, then X is polynomially convex if and only if each fiber $f^{-1}\{t\} \cap X$, $t \in \mathbb{R}$, is polynomially convex. If X is polynomially convex, then $\mathcal{P}(X) = \mathcal{C}(X)$ if and only if for each t, $\mathcal{P}(f^{-1}\{t\} \cap X) = \mathcal{C}(f^{-1}\{t\} \cap X)$.*

Here, $\mathcal{P}(X)$ denotes the uniform algebra generated by all holomorphic polynomials restricted to X.

Let S_1 and S_2 be two totally-real surfaces in $\mathbb{C}^2$ passing through the origin. Their tangent spaces at the origin are also totally real. If $T_0S_1 \cap T_0S_2 = \{0\}$, then there exist global holomorphic coordinates (z,w) with respect to which $T_0S_1 = \mathbb{R}^2$ and $T_0S_2 = M(A)$ for some real matrix A, where $A + iI$ is invertible and $M(A) := (A+iI)\mathbb{R}^2$. Here $\mathbb{R}^2 := \{(z,w) : \ \mathfrak{Im}(z) = 0 = \mathfrak{Im}(w)\}$. The reader is referred to Weinstock's paper [7] for details.

Near the origin, S_1 and S_2 will be small perturbations of $\mathbb{R}^2$ and $M(A)$ respectively. Define $S_j(\delta) := S_j \cap \overline{B(0;\delta)}$, $j = 1,2$. For sufficiently small $\delta > 0$, we have

$$
\begin{aligned}
S_1(\delta) &= \{(x + f_1(x,y), y + f_2(x,y)) : x, y \in \mathbb{R}\} \cap \overline{B(0;\delta)} \\
S_2(\delta) &= \{(A+iI)(x,y) + (g_1(x,y), g_2(x,y)) : x, y \in \mathbb{R}\} \cap \overline{B(0;\delta)},
\end{aligned}
$$

where $f_j, g_j = o(||(x,y)||)$ as $(x,y) \to 0$ are $\mathbb{C}$-valued functions, $j = 1,2$.

Since $T_0S_1 \cup T_0S_2$ is locally polynomially convex at origin, it satisfies Weinstock's criterion [7], i.e. A has no purely imaginary eigenvalue of modulus greater than 1. It is easy to show that the image of $M(A) \cup \mathbb{R}^2$ under a $\mathbb{C}$-linear transformation represented by a *real* nonsingular matrix S is $M(SAS^{-1}) \cup \mathbb{R}^2$, whence this transformation maps $S_1(\delta) \cup S_2(\delta)$ to $\widetilde{S_1}(\delta) \cup \widetilde{S_2}(\delta)$, where

$$\widetilde{S_1}(\delta) = \{(x + \widetilde{f_1}(x,y), y + \widetilde{f_2}(x,y)) : x, y \in \mathbb{R}\} \cap S(\overline{B(0;\delta)})$$
$$\widetilde{S_2}(\delta) = \{(SAS^{-1} + iI)(x,y) + (\widetilde{g_1}(x,y), \widetilde{g_2}(x,y)) : x, y \in \mathbb{R}\} \cap S(\overline{B(0;\delta)}),$$

where $\widetilde{f_j}, \widetilde{g_j}$ have the same properties as f_j, g_j given above, $j = 1,2$.

4. The proof of Theorem 1.2

Let S_1 and S_2 be two totally-real surfaces intersecting only at the origin and $T_0S_1 \cap T_0S_2 = \{0\}$. Let

$$S_1(\delta) = \{(x + f_1(x,y), y + f_2(x,y)) : x, y \in \mathbb{R}\} \cap \overline{B(0;\delta)}$$
$$S_2(\delta) = \{(A + iI)(x,y) + (g_1(x,y), g_2(x,y)) : x, y \in \mathbb{R}\} \cap \overline{B(0;\delta)},$$

where f_j, g_j are as described in Section 3. Now, it is a fact of basic linear algebra that every real 2×2 matrix is similar via a *real* nonsingular matrix, to one of the following three kinds of matrices: a diagonal matrix with real entries, a matrix of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ or of the form $\begin{pmatrix} s & -t \\ t & s \end{pmatrix}$ where $\lambda, s, t \in \mathbb{R}$. Given this fact, and the argument in the last paragraph of of Section 3, the proof of Theorem 1.2 reduces to the following two lemmas. This is because it is sufficient to take the matrix A to be one of the above form.

Lemma 4.1. *If* $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, *where* $\lambda \in \mathbb{R}$ *or* A *is a diagonal matrix with real entries, then* $S_1 \cup S_2$ *is locally polynomially convex at origin.*

Proof. We shall show that, shrinking δ if necessary, $S_1(\delta) \cup S_2(\delta)$ is polynomially convex. Consider the polynomial

$$P(z) \;=\; \langle (A - iI)z, z \rangle$$

where $\langle z, w \rangle \;:=\; z_1w_1 + z_2w_2$. We will first consider the case when A is a Jordan block.

$$\begin{aligned} P(x + f_1(x,y), y + f_2(x,y)) &= \langle ((\lambda - i)x + y, (\lambda - i)y), (x,y) \rangle + H(x,y) \\ &= ((\lambda - i)x + y)x + (\lambda - i)y^2 + H(x,y) \\ &= (\lambda - i)x^2 + xy + (\lambda - i)y^2 + H(x,y) \\ &= (\lambda x^2 + xy + \lambda y^2) - i(x^2 + y^2) + H(x,y), \end{aligned}$$

where $H(x,y) = o(||(x,y)||^2)$ as $(x,y) \to 0$.

Since $\lim_{(x,y)\to 0} H(x,y)/||(x,y)||^2 = 0$, taking $\delta > 0$ sufficiently small,

$$\mathfrak{Im}(P(z)) < 0 \;\; \forall z \in S_1(\delta) \setminus \{0\}$$

and equal to zero only when $z = 0$.

Now, for $z \in S_2(\delta)$

$$
\begin{aligned}
P(z) &= P((\lambda+i)x + y + g_1(x,y), (\lambda+i)y + g_2(x,y)) \\
&= \langle ((\lambda^2+1)x + 2\lambda y, (\lambda^2+1)y), ((\lambda+i)x + y, (\lambda+i)y)\rangle + o(||(x,y)||^2) \\
&= [(\lambda+i)x + y][(\lambda^2+1)x + 2\lambda y] + (\lambda^2+1)(\lambda+i)y^2 + o(||(x,y)||^2) \\
&= [(\lambda^2+1)\lambda x^2 + (2\lambda^2 + (\lambda^2+1))xy + (2\lambda+\lambda^3)y^2] \\
&\quad + i[(\lambda^2+1)x^2 + 2\lambda xy + (\lambda^2+1)y^2] + o(||(x,y)||^2).
\end{aligned}
$$

Here

$$
\begin{aligned}
\mathfrak{Im}(P(z)) &= (\lambda^2+1)x^2 + 2\lambda xy + (\lambda^2+1)y^2 + o(||(x,y)||^2) \\
&= \lambda^2 x^2 + y^2 + (x+\lambda y)^2 + o(||(x,y)||^2).
\end{aligned}
$$

So, shrinking $\delta > 0$ if necessary,

$$\mathfrak{Im}(P(z)) > 0 \quad \forall z \in S_2(\delta) \setminus \{0\}$$

and equal to zero only when $z = 0$.

We can now show that $P^{-1}\{0\} \cap (S_1(\delta) \cup S_2(\delta))$ is polynomially convex. Observe that $P^{-1}\{0\} \cap (S_1(\delta) \cup S_2(\delta)) = \{(0,0)\}$, hence polynomially convex. By Lemma 3.1 $S_1(\delta) \cup S_2(\delta)$ is polynomially convex. The same proof goes through with P defined exactly as above (and with considerably simpler calculations) when A is a diagonal matrix with real entries. $\square$

We now have to consider the case when $A = \begin{pmatrix} s & -t \\ t & s \end{pmatrix}$. Recall that, by hypothesis, $M(A) \cup \mathbb{R}^2$ is locally polynomially convex at $0 \in \mathbb{C}^2$. By Weinstock's criterion [7], $t \in \mathbb{R}$ will satisfy $|t| < 1$ whenever $s = 0$. It is this requirement that shapes our next lemma.

Lemma 4.2. *If $A = \begin{pmatrix} s & -t \\ t & s \end{pmatrix}$ and $|t| < 1$ whenever $s = 0$, then $S_1 \cup S_2$ is locally polynomially convex at origin.*

Proof. As before

$$
\begin{aligned}
S_1(\delta) &= \{(x + f_1(x,y), y + f_2(x,y)) : ||(x,y)|| \le \delta\} \\
S_2(\delta) &= \{((s+i)x - ty + g_1(x,y), tx + (s+i)y + g_2(x,y)) : ||(x,y)|| \le \delta\}.
\end{aligned}
$$

Consider the polynomial

$$F(z_1, z_2) = z_1^2 + z_2^2.$$

So,

$$F(x + f_1(x,y), y + f_2(x,y) = x^2 + y^2 + H_1(x,y),$$

where $H_1(x,y) = o(||(x,y)||^2)$ as $(x,y) \to 0$.

$$
\begin{aligned}
&F((s+i)x - ty + g_1(x,y), tx + (s+i)y + g_2(x,y)) \\
&\qquad = (s^2+t^2-1)x^2 + (s^2+t^2-1)y^2 + 2si(x^2+y^2) + H_2(x,y),
\end{aligned}
$$

where $H_2(x,y) = o(||(x,y)||^2)$ as $(x,y) \to 0$.

Hence, for $\delta > 0$ sufficiently small,

$$\mathfrak{Re}(F(z)) \ge 0 \ \ \forall z \in S_1(\delta)$$

and equal to zero only when $z = 0$.

Case I. *When $(s^2 + t^2) < 1$.*

Clearly, after shrinking $\delta > 0$ if necessary, $\mathfrak{Re}(F(z)) \leq 0 \ \forall \ z \in S_2(\delta)$ and equal to zero only when $z = 0$.

Case II. *When $s^2 + t^2 \geq 1$.*

First note that, by hypothesis, $s \neq 0$ in this case. We fix an $\varepsilon > 0$ sufficiently small, whose precise value will be specified later. Then, since $\lim_{(x,y)\to 0} H_2(x,y)/||(x,y)||^2 = 0$, $\exists \delta_\varepsilon > 0$ such that

$$F(S_2(\delta_\varepsilon)) \subset \{u + iv \in \mathbb{C} : |(s^2 + t^2 - 1)v - 2su| < \varepsilon|v|\}.$$

Call the set in the right hand side as $\mathfrak{C}_{2,\varepsilon}$. In fact shrinking δ_ε further if necessary we shall also get

$$F(S_1(\delta_\varepsilon) \subset \mathfrak{C}_{1,\varepsilon} := \{u + iv \in \mathbb{C} : |v| < \varepsilon u\}.$$

Now choose sufficiently small $\varepsilon_0 > 0$ such that

$$\mathfrak{C}_{1,\varepsilon_0} \cap \mathfrak{C}_{2,\varepsilon_0} \ = \ \{0\},$$

and write $\delta = \delta_{\varepsilon_0}$.

Hence, in both the cases $F(S_1(\delta))$ and $F(S_2(\delta))$ lie in two different angular sectors intersecting only at the origin. We also have

$$F^{-1}\{0\} \cap (S_1(\delta) \cup S_2(\delta)) = \{(0,0)\}$$

is polynomially convex. So, all the hypotheses of Lemma 3.1 are satisfied. Hence $S_1(\delta) \cup S_2(\delta)$ is polynomially convex. □

In view of our remarks above, Lemmas 4.1 and 4.2 give us the result. □

5. The proof of Theorem 1.4

Let P be a $\mathbb{C}$-linear function such that

$$M_1 \cup M_2 \subset \mathbb{H} := \{(z,w) \in \mathbb{C}^2 : \ \mathfrak{Im} P(z,w) = 0\}.$$

By interchanging the roles of z and w if necessary, we may assume that $\partial_w P \not\equiv 0$. Now consider the biholomorphic map $\Phi : \begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z \\ P(z,w) \end{pmatrix}$ from $\mathbb{C}^2$ to $\mathbb{C}^2$.

We have

$$\Phi(M_1 \cup M_2) \subset \mathbb{C}_z \times \mathbb{R}_u \qquad \qquad (\text{taking } w = u + iv)\,.$$

Since $span_{\mathbb{C}}\{M_1 \cap M_2\} \subset \mathbb{H}$ and $\mathbb{H}$ contains a unique complex line namely $\{(z,w) \in \mathbb{C}^2 : \ P(z,w) = 0\}$, $\Phi(span_{\mathbb{C}}\{M_1 \cap M_2\}) = \mathbb{C}_z \times \{0\}$ and hence $M_1 \cap M_2 \subset \mathbb{C}_z \times \{0\}$. Now we can find a $\theta \in [0, 2\pi)$ such that if $\Psi := (e^{i\theta}\Phi_1, \Phi_2)$, then

$$\begin{aligned} &\Psi(M_1 \cap M_2) = \{(x,0) \in \mathbb{C}^2 : \ x \in \mathbb{R}\}, \ \text{ and} \\ &\Psi(M_1 \cup M_2) \subset \mathbb{C}_z \times \mathbb{R}_u. \end{aligned}$$

Let us find the equations of $\Psi(M_j)$, $j = 1, 2$. The analysis reduces to exactly two cases.

Case I. *When neither $\Psi(M_1)$ nor $\Psi(M_2)$ is perpendicular to $\mathbb{C}_z \times \{0\}$.*

In this case $\Psi(M_1)$ and $\Psi(M_2)$ both can be written in the graph form. Writing $z = x + iy$, we get:

$$\text{Equation of } \Psi(M_j) = \begin{cases} A_j x + B_j y + D_j u &= 0, \\ \qquad\qquad\quad v &= 0, \ j = 1, 2. \end{cases}$$

Both the planes $\Psi(M_j),\ j=1,2$, pass through $\{(x,0)\in\mathbb{C}^2 : x\in\mathbb{R}\}$. Hence, $A_j=0,\ j=1,2$, and hence, there exist $C_1,C_2\in\mathbb{R}\setminus\{0\},\ C_1\neq C_2$ such that

$$\Psi(M_1)=\{(x+iy,C_1y)\in\mathbb{C}^2 :\ x+iy\in\mathbb{C}\},$$
$$\Psi(M_2)=\{(x+iy,C_2y)\in\mathbb{C}^2 :\ x+iy\in\mathbb{C}\}.$$

Now, for $\varepsilon>0$, write

$$F_j^\varepsilon:=\varepsilon x^2+\phi_j(z),$$

where ϕ_j are real valued functions with $\phi_j(z)=o(|z|^2)$, and set

$$\widetilde{S_j^\varepsilon}:=\{(x+iy,C_jy+F_j^\varepsilon(x,y)) :\ x,y\in\mathbb{R}\},\ j=1,2.$$

Consider the two parabolas in $\mathbb{C}_z$: $\mathfrak{Q}_j(\varepsilon,\delta):=\{x+iy\in\mathbb{C} :\ (y-\delta/C_j)=-(\varepsilon/C_j)x^2\}$, $j=1,2$, and the following small perturbations of the above parabolas

$$\widetilde{\mathfrak{Q}}_j(\varepsilon,\delta):=\{x+iy\in\mathbb{C} :\ C_jy+F_j^\varepsilon(x,y)=\delta\},\ j=1,2,$$

where $\delta>0$ is sufficiently small.

It is an absolutely elementary fact that $\mathbb{C}_z\setminus(\mathfrak{Q}_1(\varepsilon,\delta)\cup\mathfrak{Q}_2(\varepsilon,\delta))$ has a bounded component $\mathfrak{D}(\varepsilon,\delta)$ and hence, for each $\varepsilon>0$, there exists $\Delta_0(\varepsilon)>0$ such that $\mathbb{C}_z\setminus(\widetilde{\mathfrak{Q}}_1(\varepsilon,\delta)\cup\widetilde{\mathfrak{Q}}_2(\varepsilon,\delta))$ has a bounded component $\widetilde{\mathfrak{D}}(\varepsilon,\delta)$ for all $\delta\in(0,\Delta_0(\varepsilon))$.

Hence, $\mathfrak{A}_\delta:=\overline{\widetilde{\mathfrak{D}}(\varepsilon,\delta)}\times\{\delta\}$ are closed analytic discs with boundaries in $\widetilde{S_1^\varepsilon}\cup\widetilde{S_2^\varepsilon}$ for each $\delta\in(0,\Delta_0(\varepsilon))$ and $\mathfrak{A}_\delta\to\{0\}$ as $\delta\searrow 0$.

Clearly $\widetilde{S_1^\varepsilon}\cup\widetilde{S_2^\varepsilon}$ is not polynomially convex at the origin. Hence, $S_j^\varepsilon:=\Psi^{-1}(\widetilde{S_j^\varepsilon}),\ j=1,2$, are the required perturbations.

Case II. *When one of* $\Psi(M_j),\ j=1,2$, *is perpendicular to* $\mathbb{C}_z\times\{0\}$.
Let us assume that $\Psi(M_1)$ is perpendicular to $\mathbb{C}_z\times\{0\}$. So, $\Psi(M_2)$ can be written in graph form.

We have,

$$\Psi(M_1)=\{(x,u)\in\mathbb{C}^2 :\ x,u\in\mathbb{R}\}$$
$$\Psi(M_2)=\{(x+iy,Cy)\in\mathbb{C}^2 :\ x,y\in\mathbb{R}\}$$

where $C\in\mathbb{R}\setminus\{0\}$. Choosing F_2^ε exactly same as in *Case I* we write:

$$\widetilde{S_1^\varepsilon}:=M_1$$
$$\widetilde{S_2^\varepsilon}:=\{(x+iy,Cy+F_2^\varepsilon(x,y))\in\mathbb{C}^2 :\ x,y\in\mathbb{R}\}.$$

An analysis entirely similar to *Case I* will yield a $\Delta_0(\varepsilon)>0$ and closed analytic discs $\mathfrak{A}_\delta$ with boundaries in $\widetilde{S_1^\varepsilon}\cup\widetilde{S_2^\varepsilon}$ such that $\mathfrak{A}_\delta\to\{0\}$ as $\delta\searrow 0$.

As before, $S_j^\varepsilon:=\Psi^{-1}(\widetilde{S_j^\varepsilon}),\ j=1,2$, are the required perturbations. □

6. The proof of Theorem 1.5

Since $dim_{\mathbb{R}}(T_0S_1\cap T_0S_2)=1$, $span_{\mathbb{R}}(T_0S_1\cup T_0S_2)$ is a real three dimensional subspace of $\mathbb{C}^2$ and there exists a $\mathbb{C}$-linear map $P:\mathbb{C}^2\longrightarrow\mathbb{C}$ such that

$$span_{\mathbb{R}}(T_0S_1\cup T_0S_2)=\left\{(z,w)\in\mathbb{C}^2\ :\ \mathfrak{Im}(P((z,w))=0\right\}=:\mathbb{H}.$$

The condition $S_1\cup S_2\subset span_{\mathbb{R}}(T_0S_1\cup T_0S_2)$ implies $S_1\cup S_2\subset\mathbb{H}$. Therefore we have:

$$\widehat{(S_1(\delta)\cup S_2(\delta))}\subset cvx(S_1(\delta)\cup S_2(\delta))\subset\mathbb{H},$$

where $S_j(\delta) = S_j \cap \overline{B(0;\delta)},\ j = 1,2$ (here, $B(0;\delta)$ denotes a ball in $\mathbb{C}^2$ centred at origin and having radius δ) and $cvx(S)$ denotes the convex hull of S. We consider the biholomorphism $\Phi : \begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z \\ P(z,w) \end{pmatrix}$ (as before, we may assume, interchanging the roles of z and w if necessary, that $\partial_w P \not\equiv 0$ from $\mathbb{C}^2$ to $\mathbb{C}^2$ which has the following effect:

$$\Phi(S_1 \cap S_2) \subset \mathbb{C}_z \times \mathbb{R}_u, \quad (\text{where } z = x+iy,\ w = u+iv)$$
$$[\Phi(S_1(\delta) \cap S_2(\delta))]\widehat{\ } \subset \mathbb{C}_z \times \mathbb{R}_u.$$

Our examination involves exactly two cases. Let π_z denote the projection onto the first coordinate.

Case I. *When $\pi_z[\Phi(T_0S_1 \cap T_0S_2)]$ is a line in $\mathbb{C}_z \times \{0\}$.*

We make one final adjustment. Let θ be the angle between the line $\{(x,0) :\ x \in \mathbb{R}\}$ and $\pi_z[\Phi(T_0S_1 \cap T_0S_2)]$ in $\mathbb{C}_z \times \{0\}$, and let $\Psi := (e^{-i\theta}\Phi_1, \Phi_2)$. Note that, from the assumption $span_{\mathbb{C}}\{T_0S_1 \cap T_0S_2\} \nsubseteq\ span_{\mathbb{R}}\{T_0S_1 \cup T_0S_2\}$, we get $\Psi(T_0S_1 \cap T_0S_2)$ is not the x-axis. Hence $\exists a \in \mathbb{R} \setminus \{0\}$ such that

$$\Psi(T_0S_1 \cap T_0S_2) \quad : y = 0,\ v = 0,\ u = ax.$$

Furthermore, we have:

$$\text{Equation of } \Psi(T_0S_j) = \begin{cases} u &= ax + B_j y, \\ v &= 0,\ B_j \in \mathbb{R},\ j = 1,2, \text{ and } B_1 \neq B_2. \end{cases}$$

For sufficiently small $\delta > 0$,

$$\widetilde{S_j}(\delta) := \Psi(S_j) \cap \overline{B(0,\delta)} = \begin{cases} u &= ax + B_j y + \varphi_j(x,y), \\ v &= 0, \end{cases}$$

where $\varphi_j(x,y) = O(|(x,y)|^2),\ j = 1,2$. We consider the polynomial $f(z,w) = w$. There is a small neighbourhood $\omega(\delta)$ of $0 \in \mathbb{C}_z$ such that

$$f^{-1}\{t\} \cap (\widetilde{S_1}(\delta) \cup \widetilde{S_2}(\delta)) = \mathcal{K}_1^t \cup \mathcal{K}_2^t$$

where

$$\mathcal{K}_1^t := \{(x+iy,t)\ :\ ax + B_1 y + \varphi_1(x,y) = t, (x,y) \in \omega(\delta)\},$$
$$\mathcal{K}_2^t := \{(x+iy,t)\ :\ ax + B_2 y + \varphi_2(x,y) = t, (x,y) \in \omega(\delta)\}.$$

Note that if $\varphi_1 = 0$ and $\varphi_2 = 0$, then the above union would have been a union of two line segments, which is polynomially convex. Without loss of generality, we may take $B_1 \neq 0$. Then, $\pi_z(\mathcal{K}_1^t)$ is the graph of the function $\psi_1 : \omega(\delta) \cap \mathbb{R}_x \longrightarrow \mathbb{R}$ with $\frac{d\psi_1}{dx}(0) = -a/B_1$. On the other hand:

$$\pi_z(\mathcal{K}_2^t) = \begin{cases} \text{the graph of a function } \psi_2 : \omega(\delta) \cap \mathbb{R}_y \longrightarrow \mathbb{R}, \text{ if } B_2 = 0, \\ \text{the graph of a function } \widetilde{\psi_2} : \omega(\delta) \cap \mathbb{R}_x \longrightarrow \mathbb{R} \text{ with } \frac{d\widetilde{\psi_2}}{dx}(0) = -a/B_2, \\ \hfill \text{if } B_2 \neq 0, \end{cases}$$

for $\omega(\delta)$ sufficiently small, and viewing $\mathbb{C} \cong \mathbb{R}_x \times \mathbb{R}_y$. Here is a brief justification of the above descriptions of $\pi_z(\mathcal{K}_j^t),\ j = 1,2$. Note that the equation $ax + \varphi_2(x,0) = t$ will have a unique solution in $\omega(\delta) \cap \mathbb{R}_x$, say $x = x_0(t)$, once we have chosen a $\delta > 0$ sufficiently small and fixed it, for all $t \in \mathbb{R}$ approaching to 0. By the Implicit Function

Theorem, ψ_2 is a function satisfying $\psi_2(0) = x_0(t)$ and $\frac{d\psi_2}{dy}(0) = -\partial_y\varphi_2(x_0(t),0)/(a+\partial_x\varphi_2(x_0(t),0))$. A similar, but easier, argument gives the descriptions of ψ_1 and $\widetilde{\psi_2}$.

In either case, $\pi_z(\mathcal{K}_1^t)\cap\pi_z(\mathcal{K}_2^t)$ does not separate $\mathbb{C}_z\times\{0\}$, whence $\mathcal{K}_1^t\cup\mathcal{K}_2^t$ does not separate $\mathbb{C}_z\times\{t\}$, provided we choose and fix $\delta>0$ sufficiently small. In view of Result 3.2, $\widetilde{S_1}(\delta)\cup\widetilde{S_2}(\delta)$ is polynomially convex. As Ψ is a biholomorphism, we infer that $S_1\cup S_2$ is locally polynomially convex at $(0,0)\in\mathbb{C}^2$.

Case II. *When* $\Phi(T_0S_1\cap T_0S_2)=\{(0,u)\in\mathbb{C}^2\ :\ u\in\mathbb{R}\}$.
Since $\Phi(T_0S_1\cap T_0S_2)=\{(0,u)\in\mathbb{C}^2\ :\ u\in\mathbb{R}\}$, both the planes T_0S_1 and T_0S_2 are perpendicular to $\mathbb{C}_z\times\{0\}$ in $\mathbb{C}^2$. We can find an angle θ such that if we define $\Psi(z,w):=(e^{i\theta}\Phi_1,\Phi_2)$ then, neither $\pi_z\circ\Psi(T_0S_1)$ nor $\pi_z\circ\Psi(T_0S_2)$ is the x-axis or the y-axis. Hence we have:

$$\text{Equation of }\Psi(T_0S_j)=\begin{cases} y &= A_jx,\\ v &= 0,\ A_j\in\mathbb{R}\setminus\{0\},\ j=1,2,\ \text{and } A_1\neq A_2.\end{cases}$$

For sufficiently small $\delta>0$,

$$\widetilde{S_j}(\delta):=\Psi(S_j)\cap\overline{B(0,\delta)}=\begin{cases} y &= A_jx+\varphi_j(x,u),\\ v &= 0,\end{cases}$$

where $\varphi_j(x,u)=O(|(x,u)|^2)$, $j=1,2$. As in the first case, we consider the polynomial $f(z,w)=w$. There is a small neighbourhood ω of $0\in\mathbb{C}_z$ such that

$$f^{-1}\{t\}\cap(\widetilde{S_1}(\delta)\cup\widetilde{S_2}(\delta))=\mathcal{K}_1^t\cup\mathcal{K}_2^t$$

where

$$\mathcal{K}_1^t:=\{(x+iy,t)\ :\ y=A_1x+\varphi_1(x,t),(x,y)\in\omega(\delta)\}$$
$$\mathcal{K}_2^t:=\{(x+iy,t)\ :\ y=A_2x+\varphi_2(x,t),(x,y)\in\omega(\delta)\}.$$

Note here also that if $\varphi_1=0$ and $\varphi_2=0$, then the above union would have been a union of two line segments, which is polynomially convex. In this case $\pi_z(\mathcal{K}_j^t)$ is the graph of the function $\psi_j:\omega(\delta)\cap\mathbb{R}_x\longrightarrow\mathbb{R}$ with $\frac{d\psi_j}{dx}(0)=A_j$ for $j=1,2$. Hence, $\mathcal{K}_1^t\cup\mathcal{K}_2^t$ does not separate $\mathbb{C}_z\times\{t\}$, provided we choose and fix $\delta>0$ sufficiently small. Hence, in view of Result 3.2, $\widetilde{S_1}(\delta)\cup\widetilde{S_2}(\delta)$ is polynomially convex. As Ψ is a biholomorphism, we infer that $S_1\cup S_2$ is locally polynomially convex at $(0,0)\in\mathbb{C}^2$. $\square$

7. The Proof of Theorem 1.8

We begin by observing that since $T_0S_1\neq T_0S_2,\ \lambda\neq 0$. We shall use Kallin's lemma with the following polynomial

$$P(z,w)=z+w+\overline{\alpha}z^2+\alpha w^2,$$

where $\alpha\in\mathbb{C}$ will be chosen suitably. First, we examine the image of $S_1\cap U$ under P. Let us designate

$$\phi_1(z):=\overline{A}z^2+A\overline{z}^2+C_1z\overline{z}+O(|z|^3),\ \ z\in D(0;\delta).$$

Thus we have,

$$\begin{aligned} P(z,\overline{z}+\phi_1(z)) &= z+\overline{z}+A\overline{z}^2+\overline{A}z^2+C_1|z|^2+\overline{\alpha}z^2+\alpha\overline{z}^2+O(|z|^3),\\ \Im\mathfrak{m}P(z,\overline{z}+\phi_1(z)) &= \Im\mathfrak{m}C_1|z|^2+O(|z|^3)\ \ \forall z\in D(0;\delta). \end{aligned} \tag{7.1}$$

Consequently, by using condition (i), we can find a $\delta_1 \in (0, \delta)$ sufficiently small so that

$$P^{-1}\{0\} \cap S_1(\delta_1) = \{0\}, \tag{7.2}$$

where $S_1(\delta_1) = S_1 \cap \overline{D(0;\delta_1)} \times \mathbb{C}$.

Now let us look at the image of $S_2(\delta_1)$ $(= S_2 \cap \overline{D(0;\delta_1)} \times \mathbb{C})$ under the polynomial P.

$$\begin{aligned}
&P(z, \overline{z} + \lambda\overline{z} + \overline{\lambda}z + \phi_2(z)) \\
&= z + \overline{z} + \lambda\overline{z} + \overline{\lambda}z + \phi_2(z) + \overline{\alpha}z^2 + \alpha(\overline{z} + \lambda\overline{z} + \overline{\lambda}z)^2 + O(|z|^3) \\
&= z + \overline{z} + \lambda\overline{z} + \overline{\lambda}z + A_2z^2 + B_2\overline{z}^2 + C_2|z|^2 + \overline{\alpha}z^2 + \alpha\overline{z}^2 + 2\alpha\overline{z}(\lambda\overline{z} + \overline{\lambda}z) \\
&\qquad\qquad + \alpha(\lambda\overline{z} + \overline{\lambda}z)^2 + O(|z|^3) \\
&= z + \overline{z} + \lambda\overline{z} + \overline{\lambda}z + A_2z^2 + (B_2 + 2\alpha\lambda)\overline{z}^2 + (C_2 + 2\alpha\overline{\lambda})|z|^2 + \overline{\alpha}z^2 + \alpha\overline{z}^2 \\
&\qquad\qquad + \alpha(\lambda\overline{z} + \overline{\lambda}z)^2 + O(|z|^3)
\end{aligned}$$

We choose α such that

$$\overline{A_2} = B_2 + 2\alpha\lambda.$$

Note that

$$C_2 + 2\alpha\overline{\lambda} = C_2 + \frac{(\overline{A_2} - B_2)\overline{\lambda}^2}{|\lambda|^2},$$

and observe:

$$\begin{aligned}
\Im\mathfrak{m}(P(z, \overline{z} + \overline{\lambda}z + \lambda\overline{z} + \phi_2(z))) &= \Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}^2}{|\lambda|^2} + C_2\right)|z|^2 \\
&+ \Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}}{2|\lambda|^2}\right)(\overline{\lambda}z + \lambda\overline{z})^2 + O(|z|^3) \;\; \forall z \in D(0;\delta).
\end{aligned} \tag{7.3}$$

We examine the second term on the right hand side of (7.3):

$$\Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}}{2|\lambda|^2}\right) \geq 0 \Longrightarrow \Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}}{2|\lambda|^2}\right)(\overline{\lambda}z + \lambda\overline{z})^2 \leq 2\Im\mathfrak{m}((\overline{A_2} - B_2)\overline{\lambda})|z|^2, \tag{7.4}$$

and

$$\Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}}{2|\lambda|^2}\right) < 0 \Longrightarrow \Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}}{2|\lambda|^2}\right)(\overline{\lambda}z + \lambda\overline{z})^2 \geq 2\Im\mathfrak{m}((\overline{A_2} - B_2)\overline{\lambda})|z|^2. \tag{7.5}$$

We shall divide the remaining part of the proof into two cases.

Case I. *We consider the case when* $\Im\mathfrak{m}(C_1) < 0$.
So, $\text{sgn}(\Im\mathfrak{m}(C_1)) = -1$ and hence by condition (i)

$$\Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}^2}{|\lambda|^2} + C_2\right) > 0.$$

If $\Im\mathfrak{m}((\overline{A_2} - B_2)\overline{\lambda}) \geq 0$ then by (7.4) there is a $\delta_2 \in (0, \delta_1)$ such that when $z \in D(0;\delta_2)$,

$$\Im\mathfrak{m}(P(z, \overline{z} + \overline{\lambda}z + \lambda\overline{z} + \phi_2(z))) \geq 0 \tag{7.6}$$

and equalling 0 if and only if $z = 0$. On the other hand, if $\Im\mathfrak{m}((\overline{A_2} - B_2)\overline{\lambda}) < 0$, then by (7.5)

$$\begin{aligned}
&\Im\mathfrak{m}\big(P(z, \overline{z} + \overline{\lambda}z + \lambda\overline{z} + \phi_2(z))\big)\\
&\geq \Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}^2}{|\lambda|^2} + C_2\right)|z|^2 + 2\Im\mathfrak{m}((\overline{A_2} - B_2)\overline{\lambda})|z|^2 + O(|z|^3)\\
&= \left(\Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}^2}{|\lambda|^2} + C_2\right) + 2\Im\mathfrak{m}((\overline{A_2} - B_2)\overline{\lambda})\right)|z|^2 + O(|z|^3).
\end{aligned}$$

Hence by condition (ii), and arguing exactly as above, we get that there is a $\delta_2 \in (0, \delta_1)$ such that when $z \in D(0; \delta_2)$,

$$\Im\mathfrak{m}(P(z, \overline{z} + \overline{\lambda}z + \lambda\overline{z} + \phi_2(z))) \geq 0 \tag{7.7}$$

and equalling 0 if and only if $z = 0$.

Hence from (7.1),(7.6) and (7.7), we have the following:

There exists $\delta_2 > 0$ *such that*

- $P^{-1}\{0\} \cap S_2(\delta_2) = \{0\}$; *and*
- $P(S_1(\delta_2))$ *and* $P(S_2(\delta_2))$ *lie in the lower and upper half planes respectively and intersect only at the origin.*

Case II. *We consider the case when* $\Im\mathfrak{m}(C_1) > 0$.
Then by condition (i),

$$\Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}^2}{|\lambda|^2} + C_2\right) < 0.$$

We argue similarly as in case *Case I.* If $\Im\mathfrak{m}((\overline{A_2} - B_2)\overline{\lambda}) < 0$ then there is a $\delta_2 \in (0, \delta_1)$ such that when $z \in D(0; \delta_2)$,

$$\Im\mathfrak{m}(P(z, \overline{z} + \overline{\lambda}z + \lambda\overline{z} + \phi_2(z))) \leq 0 \tag{7.8}$$

and equalling 0 if and only if $z = 0$. On the other hand, if $\Im\mathfrak{m}((\overline{A_2} - B_2)\overline{\lambda}) \geq 0$, then:

$$\begin{aligned}
&\Im\mathfrak{m}\big(P(z, \overline{z} + \overline{\lambda}z + \lambda\overline{z} + \phi_2(z))\big)\\
&\leq -\left|\Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}^2}{|\lambda|^2} + C_2\right)\right||z|^2 + 2\Im\mathfrak{m}((\overline{A_2} - B_2)\overline{\lambda})|z|^2 + O(|z|^3)\\
&= \left(-\left|\Im\mathfrak{m}\left(\frac{(\overline{A_2} - B_2)\overline{\lambda}^2}{|\lambda|^2} + C_2\right)\right| + 2\Im\mathfrak{m}\left((\overline{A_2} - B_2)\overline{\lambda}\right)\right)|z|^2 + O(|z|^3).
\end{aligned} \tag{7.9}$$

Hence by condition (ii) and (7.9), there is a $\delta_2 \in (0, \delta)$ such that when $z \in D(0; \delta_2)$,

$$\Im\mathfrak{m}(P(z, \overline{z} + \overline{\lambda}z + \lambda\overline{z} + \phi_2(z))) \leq 0 \tag{7.10}$$

and equalling 0 if and only if $z = 0$.

In this case also from (7.1), (7.8) and (7.10) we have the following:

There exists $\delta_2 > 0$ *such that*

- $P^{-1}\{0\} \cap S_2(\delta_2) = \{0\}$; *and*
- $P(S_1(\delta_2))$ *and* $P(S_2(\delta_2))$ *lie in the upper and lower half planes respectively and intersect only at the origin.*

Therefore, with this choice of P, in all cases, the hypotheses of Kallin's lemma (Lemma 3.1) are met and hence $S_1(\delta_2) \cup S_2(\delta_2)$ is polynomially convex. Hence $S_1 \cup S_2$ is locally polynomially convex at the origin. $\square$

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